INTEGRATION FORMULAE & ITS TYPES

INTRODUCTION:-

- The name integral calculus has its origin in the process if summation because the word **"To Integrate"** means **"To find the sum of"**
- These generalizations of integrals were developed from the need of physics & these credit goes to Sir Isaac Newton & Gottfried Leibnitz. The definition of integral was given by **"Bernhord Riemann".**

Definition:-

If $\varnothing(n)$ is continuous function of x then there exist a function f(x) such that $f'(x)=\varnothing(x)$. We call f(x) is an integral of $\varnothing(x)$ and operation of finding integral is called as integration.

In Symbolic form, we write

$$\int \varnothing(x).dx=f(x) \quad \text{or} \quad f'(x).dx = f(x)$$

- Integration is the Inverse Process of Derivative.

In these chapter we will study about "indefinite integral".

Thus, $\int\varnothing(x).dx=f(x)+c$ where c = arbitrary constant

ALGEBRA OF INTEGRATION

1. $\int[f(x)\pm g(x)]dx=\int f(x).dx\pm\int g(x)dx$
2. $\int k.f(x)dx=k\int f(x).dx,\ \ k\neq0$ constant
3. $\int f(ax+b).dx=\frac{1}{a}\varnothing(ax+b)+c \quad \text{if} \int f(x).dx=\varnothing(x)+c$

Note:-

In integration we adjust the derivative of x in denominator.

Formulae:-

1. $\int 1.dx = x + c$
2. $\int k.dx = kx + c$
3. $\int x^{n}.dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq 1$
4. $\int x.dx = \frac{x^{2}}{2} + c$
5. $\int x^{2}.dx = \frac{x^{3}}{3} + c$
6. $\int \frac{1}{x^{2}}.dx = -\frac{1}{x} + c$
7. $\int \frac{1}{x}.dx = \log x + c$
8. $\int \frac{1}{\sqrt{x}}.dx = 2\sqrt{x} + c$
9. $\int \frac{1}{x\sqrt{x}}.dx = \frac{-2}{\sqrt{x}} + c$
10. $\int a^{x}.dx = \frac{a^{x}}{\log a} + c$
11. $\int e^{x}.dx = e^{x} + c$
12. $\int \log x.dx = x\log x - x + c$
13. $\int \sin x.dx = -\cos x + c$
14. $\int \cos x.dx = \sin x + c$
15. $\int \tan x.dx = \log|\sec x| + c$
16. $\int \cot x.dx = \log|\sin x| + c$
17. $\int \sec x.dx = \log|\sec x + \tan x| + c$

    $= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

18. $\int \text{cosec}x.dx = \log|\text{cosec}x - \cot x| + c$

$= \log|\tan(\frac{x}{2})| + c$

19. $\int \sec^2 x.dx = \tan x + c$

20. $\int \text{cosec}^2 x.dx = -\cot x + c$

21. $\int \sec x.\tan x.dx = \sec x + c$

22. $\int \text{cosec}x.\cot x.dx = -\text{cosec}x + c$

23. $\int \sin^3 x.dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c$

24. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c = -\cos^{-1}x + c$

25. $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c = -\cot^{-1}x + c$

26. $\int \frac{1}{x\sqrt{1-x^2}} dx = \sec^{-1}x + c = -\text{cosec}^{-1}x + c$

27. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

28. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

29. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

30. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

31. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right| + c$

32. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + c$

33. $\int \frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^2\left(\frac{x}{a}\right)+c$

34. $\int \sqrt{a^2+x^2}.dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+c$

35. $\int \sqrt{a^2-x^2}.dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left|\frac{x}{a}\right|+c$

36. $\int \sqrt{x^2-a^2}.dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c$

Methods of Integration

- Integration by Substitution
- Integration by Parts
- Integration by Partial Fractions

Integration by Substitution:-

In this method we reduce the given function to standard form by changing variable x to t, using some suitable substitution, x = ∅(t)

Theorem:-

If x= ∅(t) if differentiable function of t, then $\int f(x).dx=\int f[\varnothing(t)]\varnothing'(t)dt$.

- Proof may be expected in exam Ref Page No.122.
- Take tge function & put the fuction as 't'.
- Then take the derivative of the function.
- Then rearrange the function in variable 't' & apply the formula for suitable function.
- Re-substitute the value of t.

Corollary 1:-

$$\int [f(x)]^n f'(x).dx=\frac{[f(x)]^{n+1}}{n+1}+c$$

Corollary 2:-

$$\int \frac{f'(X)}{f(x)}.dx = \log|f(x)| + c$$

Corollary 3:-

$$\int \frac{f'(X)}{\sqrt{f(x)}}.dx = 2\sqrt{f(x)} + c$$

Special Case:-

1. $\frac{\sin(x+a)}{\cos(x-b)}, \frac{\cos x}{\cos(x+a)}, \frac{\sin(x-a)}{\sin(x+a)}, \frac{1}{\cos(x-a).\cos(x-b)}$ & many more cases

- Arrange the Numerator in such a way that the Numerator is look like same as Denominator.

E.g.:

1. $$\frac{\sin(x+a)}{\cos(x-b)} = \frac{\sin[(x-b)+(a+b)]}{\cos(x-b)} = \frac{\sin(x-b).\cos(a+b)+\cos(a-b).\sin(a+b)}{\cos(x-b)}$$

2. $$\frac{1}{\sin(x-a).\cos(x-b)} = \frac{1}{\cos(a-b)} X \frac{\cos(a-b)}{\sin(x-a).\cos(x-b)}$$

$$= \frac{1}{\cos(a-b)} X \frac{\cos[(x-b)-(x-a)]}{\sin(x-a).\cos(x-b)}$$

$$= \frac{1}{\cos(a-b)} X \frac{\cos(x-b).\cos(x-a)+\sin(x-b).\sin(x-a)}{\sin(x-a).\cos(x-b)}$$

- Then apply Trigo Formulae [Factorization Formula] & Expand the Function.
- Then Individual divide the Numerator by Denominator & Simplify the function & apply Integration.

2. Integral of the form,

$$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}.dx; \int \frac{ae^x+b}{ce^x+d}.dx$$

- To evaluate this type of integral, we express Nume

$$A(\text{Denominator}) + B\frac{d}{dx}(\text{Denominator})$$

- Then find the value of A & B. By comparing on both side.
- Then substitute the value of A & B in appropriate Equation & solve the Integration by Particular Method.

NOTE:

$$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}.dx; \int \frac{ae^x+b}{ce^x+d}.dx$$

- Express in (NU)=$A(DE) + B\frac{d}{dx}(DE)$, then find the value of A & B.
- Then the answer is given by in following way

=A+Blog|csinx+dcosx|+c′

OR

=A+Blog|ce^x+d|+c′

IMPORTANT SUBSITUTION:

Sr. No.	Expression	Substitution
1	$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
2	$\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
3	$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\text{cosec}\theta$
4	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos2\theta$ or $x = a\sin2\theta$
5	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x^2 = a^2\cos2\theta$ or $x^2 = a^2\sin2\theta$
6	$\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

7	$\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
8	$\sqrt{(x-a)(b-x)}$ or $\sqrt{\frac{x-a}{b-x}}$	$x = a\cos^2\theta + b\sin^2\theta$

3. Integral of the type $\int\frac{1}{ax^2+bx+c}dx$; $\int\frac{1}{\sqrt{ax^2+bx+c}}.dx$ in order to find this type of integral we may use the following steps:

- Make the coefficient of x^2 unity if it is not as,

$$\frac{1}{a}\int\frac{1}{x^2+\frac{b}{a}x+\frac{c}{a}}.dx; \qquad \frac{1}{\sqrt{a}}\int\frac{1}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}.dx$$

- Find the third term of the quadratic Equation i.e.

Third term=$[\frac{1}{2}\text{ X coefficient of x}]^2$

- Then Add & Substract the third tem in the quadratic Equation to make it complete square.
- Then use suitable formula for evaluation.

4. Integrals of the terms $\int\frac{px+q}{ax^2+bx+c}.dx$ & $\int\frac{px+q}{\sqrt{ax^2+bx+c}}.dx$

In order to find this type of integrals, we use the following steps:

- Write the Numerator $px + q$ in the form

$$px + q = A\frac{d}{dx}(ax^2+bx+c) + B$$

$$= A(2ax+b) + B$$

- Obtain the value of A & B by equating the Equation's.

- Replace $px + q = A(2ax + b) + B$ in the given integrals.

i.e. $\int \frac{px+q}{ax^2+bx+c}dx = A\int \frac{2ax+b}{ax^2+bx+c}dx + B\int \frac{1}{ax^2+bx+c}dx$

$$= A\log|ax^2+bx+c| + B\int \frac{1}{ax^2+bx+c}dx$$

OR $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + b\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

$$= 2A\sqrt{ax^2+bx+c} + B\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

NOTE:- If power of **DENOMINATOR** is equal or power of **NUMERATOR** is equal to OR greater than power of **DENOMINATOR** then divide **NUMERATOR** by **DENOMINATOR.**

5. Integrals of the form $\int \frac{1}{a+\sin^2 x}dx$; $\int \frac{1}{a+b\cos^2 x}dx$; $\int \frac{1}{a\sin^2 x+b\cos^2 s+c}dx$

   To evaluate this type of integrals we use the following steps:

   - Divide Numerator & Denomenator both by $\cos^2 x$ or $\sin^2 x$.
   - Replace $\sec^2 x$ or $\mathrm{cosec}^2 x$, if any, in denominator, by $1+\tan^2 x$ or $1+\cot^2 x$
   - Put $\tan x = t$ or $\cot x = t$

     This substitution reduces the integrals in the form $\int \frac{1}{at^2+bt+c}dt$ & evaluate the integrals.

6. Integrals of forms $\int \frac{1}{a+b\sin x}dx$; $\int \frac{1}{a+b\cos x}dx$

   $$\int \frac{1}{a\sin x+b\cos x}dx; \quad \int \frac{1}{a+b\sin x+c\cos x}dx$$

   To evaluate this type of integrals, we use the substitution

$\tan\left(\frac{x}{2}\right) = t$ then differentiate w.r.t.x.

$$\therefore \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$$

$$\therefore dx = \frac{2dt}{\sec^2(\frac{x}{2})} = \frac{2dt}{1+\tan^2(\frac{x}{2})}$$

$$\therefore dx = \frac{2dt}{1+t^2}$$

Then we have, $\sin x = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{2dt}{1+t^2}$

& $\cos x = \frac{1-\tan 2^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-t^2}{1+t^2}$

7. Integrals of forms $\int\frac{1}{a+b\sin 2x}dx$; $\int\frac{1}{a+b\cos 2x}dx$

$$\int\frac{1}{a\sin 2x+b\cos 2x}dx; \quad \int\frac{1}{a+b\sin 2x+c\cos 2x}dx$$

To evaluate this type of integrals we use substitution.

$$\tan x = t$$

differatiate w.r.t.x

$$\sec^2 x.dx = dt$$

$$\therefore dx = \frac{dt}{\sec^2 x} = \frac{dt}{1+\tan^2 x} = \frac{dt}{1+t^2}$$

Then,

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-t^2}{1+t^2}$$

2. Integration by Parts:-

The Methods of Integration by Parts is used when the integrated can be expressed as product of two suitable functions, one of which can be differentiable and the other can be integrated.

Theorem:- If u & v are two functions of x then,

$\int u.v dx = \int v.dx \frac{du}{dx}.dx$ or

$\int u.v dx = uv_1 - u^1 v_{11} + u^{11} v_{11i} - u^{111} v_{iv} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + c$

where v_1 *= integration of v*

u^1 *= integration of u*

***** Perform these unless & until the derivative of u become*

1. [Proof is expected in exam ref. Page No.160]

1. When the integrated is product of two functions out of which second has to be integrated (whose integration is known). Hence we should make the proper choice of the first functions u & second function v.
2. We can choose the first functions as the function which comes first in serial order of letter of the **"LIATE"** where

L : Logarithmic Function

I : Inverse Trigonometric Function

A : Algebraic Function

T : Trigonometric Function

E : Exponential Function

3. If the integrand contains a logarithmic or an inverse trigonometric function, take it as the first function. In all such cases, if the second function is not given, take it as 1.

1. Integral of types :- $\int e^{x}[f(x)+f'(x)]dx$

   Theorem:- $\int e^{x}[f(x)+f'(x)]dx = e^{x}f(x) + c$

   - Proof may be expected in exam Ref. Page No.164.
   - Make the function in such a way that the function is split into two parts one is an another function & other is a derivative of that function.
   - And make the function in the above form & apply the above Rule.

2. Integral of type: $\int\sqrt{a^{2}+x^{2}}.dx; \int\sqrt{a^{2}-x^{2}}.dx; \int\sqrt{x^{2}-a^{2}}.dx$

   1) $\int\sqrt{a^{2}+x^{2}}.dx = \frac{x}{\propto}\int\sqrt{a^{2}+x^{2}} + \frac{a^{2}}{2}\log|x + \sqrt{a^{2}+x^{2}} + c$

   2) $\int\sqrt{a^{2}-x^{2}}.dx = \frac{x}{\propto}\int\sqrt{a^{2}+x^{2}} + \frac{a^{2}}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

   3) $\int\sqrt{x^{2}-a^{2}}.dx = \frac{x}{2}\int\sqrt{x^{2}-a^{2}} - \frac{a^{2}}{2}\log|x + \sqrt{x^{2}-a^{2}} + c$

   Proof May be Expected in exam Refer Page No.166.

- In order to evaluate integrals of form $\int\sqrt{ax^2-bx+c}.dx$, we follow following steps:
  - Make coefficient of x^2 as unity by taking $\sqrt{a}$ common
  - Find third term & Add & Subtract third term in Quadratic Equation.
  - Then make it complete square & apply suitable formula for evaluation.

3. Integrals of the sum $\int(px+q)\sqrt{ax^2+bx+c}.dx$

- In Order to evaluate this type of integrals, write $Px + q = A\frac{d}{dx}(ax^2+bx+c) + B$
- Then find the value of A & B by equating by Equation.
- Then Put the value of A & B in appropriate Equation & make it a suitable from then apply formula according to the formula & evaluate the integrals.

4. Integration by Partial Fraction:-

If f(x) & g(x) are two polynomials then $\frac{f(x)}{g(x)}$, g(x)≠0 is called a rational algebraic function.

1) If degree of f(x)<degree of g(x) then $\frac{f(x)}{g(x)}$ is called a proper rational function.

2) If degree of f(x)≥degree of g(x) then divide f(x) by g(x) and express the function $\frac{f(x)}{g(x)}$ as

$\frac{f(x)}{g(x)} = \text{Quotient} + \frac{\text{Remainder}}{g(x)}$, g≠0

Then $\frac{\text{Remainder}}{g(x)}$ is a proper function.

Some important formula:-

Sr. No.	Rational Form	Partial Form
1	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
3	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
4	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5	$\frac{px^2+qx+r}{(x-a)^3(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$
6	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$ Where $x^2 + bx + c$ can't be factorised

- After above process make the denominator in factor formula then apply the partial form according to the above table.
- Then find the value of A & B by outing value of x or by comparing both side.
- Substitute the value of A & B in appropriate Equation then apply particular formula ccording to the integral forms.